DEFORMATIONS OF SOME COLOR LIE SUPERALGEBRAS

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Abstract. In this work infinitesimal deformations of the model filiform $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebra have been studied. All the filiform $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebras can be obtained by means of infinitesimal deformations, hence the importance of these. Thus, in particular, we give a family of filiform $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebras via linearly integrable deformations.

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1. Introduction

A well known use of the generalizations of Lie theory corresponds to the study of symmetries into physics. These symmetries are not limited to the geometrical ones of space-time. Thus, among others, the generalization of Lie theory that has been proven to be physically relevant are color Lie (super)algebras [2], [12], [13] and [14].

We shall consider color Lie superalgebras with a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading vector space due to the great amount of physical applications of this vector space, see e.g. [3], [4], [5] and [6]. In particular, we will focus our study in a very important type of nilpotent $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebra, i.e. filiform $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebras.

Filiform Lie algebras was firstly introduced in [15] by Vergne. This type of nilpotent Lie algebra has important properties; in particular, every filiform Lie algebra can be obtained by a deformation of the model filiform algebra L_n. In the same way as filiform Lie algebras, all filiform Lie superalgebras can be obtained by infinitesimal deformations of the model Lie superalgebra $L^{n,m}$ [1], [7], [8] and [9]. In [10] we generalized this concept obtaining *filiform (G, β)-color Lie superalgebras* and the model *filiform (G, β)-color Lie superalgebra* as well as the existence of "adapted" basis for these color Lie superalgebras.

In [10] we too proved that in order to obtain all the class of filiform (G, β)-color Lie superalgebras it is only necessary to determine some infinitesimal deformations of the model filiform (G, β)-color Lie superalgebra.

In this paper we have studied these infinitesimal deformations for the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ obtaining, in particular a decompostion into 10 subspaces that depend on the election of the commutation factor β. We shall focus our study in those deformations that are linearly integrable as if φ is an integrable deformation then the law $L + \varphi$ will be a filiform $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebra. Thus, in the

last section we obtain all a family of filiform $\mathbb{Z}_2 \times \mathbb{Z}_2$-color Lie superalgebras via integrable deformations.

We will not suppose any prior knowledge of the theory of Lie superalgebras. However, we do assume that the reader is familiar with the standard theory of Lie algebras. All the vector spaces that appear in this paper (and thus, all the algebras) are assumed to be $\mathbb{F}$-vector spaces ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) with finite dimension.

2. Preliminaries

We consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. the abelian group $G = \{e = (\overline{0}, \overline{0}), a = (\overline{0}, \overline{1}), b = (\overline{1}, \overline{0}), c = (\overline{1}, \overline{1})\}$ with identity element $e = (\overline{0}, \overline{0})$ and $a + a = b + b = c + c = e$, $a + b = c$, $a + c = b$, $b + c = a$.

The vector space V is said to be $\mathbb{Z}_2 \times \mathbb{Z}_2-$graded if it admits a decomposition in direct sum, $V = V_e \oplus V_a \oplus V_b \oplus V_c$. An element X of V is called homogeneous of degree γ ($deg(X) = d(X) = \gamma$), $\gamma \in \mathbb{Z}_2 \times \mathbb{Z}_2$, if it is an element of V_γ.

Let $V = V_e \oplus V_a \oplus V_b \oplus V_c$ and $W = W_e \oplus W_a \oplus W_b \oplus W_c$ be two $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded vector spaces. A linear mapping $f : V \longrightarrow W$ is said to be homogeneous of degree γ ($deg(f) = d(f) = \gamma$), $\gamma \in \mathbb{Z}_2 \times \mathbb{Z}_2$, if $f(V_\alpha) \subset W_{\alpha+\gamma}$ for all $\alpha \in \mathbb{Z}_2 \times \mathbb{Z}_2$. The mapping f is called a homomorphism of the $\mathbb{Z}_2 \times \mathbb{Z}_2-$graded vector space V into the $\mathbb{Z}_2 \times \mathbb{Z}_2-$graded vector space W if f is homogeneous of degree $e = (\overline{0}, \overline{0})$. Now it is evident how we define an isomorphism or an automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2-$graded vector spaces.

A superalgebra $\mathfrak{g}$ is just a $\mathbb{Z}_2-$graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. That is, if we denote by [,] the bracket product of $\mathfrak{g}$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta(mod 2)}$ for all $\alpha, \beta \in \mathbb{Z}_2$.

Definition 2.1. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a superalgebra whose multiplication is denoted by the bracket product [,]. We call $\mathfrak{g}$ a **Lie superalgebra** if the multiplication satisfies the following identities:

1. $[X, Y] = -(-1)^{\alpha\cdot\beta}[Y, X] \qquad \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta$.
2. $(-1)^{\gamma\cdot\alpha}[X, [Y, Z]] + (-1)^{\alpha\cdot\beta}[Y, [Z, X]] + (-1)^{\beta\cdot\gamma}[Z, [X, Y]] = 0$
   for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, Z \in \mathfrak{g}_\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

Identity 2 is called the graded Jacobi identity and it will be denoted by $J_g(X, Y, Z)$.

We observe that if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra, we have that $\mathfrak{g}_0$ is a Lie algebra and $\mathfrak{g}_1$ has structure of $\mathfrak{g}_0-$module.

Color Lie (super)algebras can be seen as a direct generalization of Lie (super)algebras. Indeed, the latter are defined through antisymmetric (commutator) or symmetric (anticommutator) products, although for the former the product is neither symmetric nor antisymmetric and is defined by means of a commutation factor. This commutation factor is equal to ± 1 for (super)Lie algebras and more general for arbitrary color Lie (super)algebras. As happened for Lie superalgebras, the basic tool to define color Lie (super)algebras is a grading determined by an Abelian group.

Definition 2.2. Let G be an Abelian group . A **commutation factor** β is a map $\beta : G \times G \longrightarrow \mathbb{F} \setminus \{0\}$, ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$), satisfying the following constraints:

(1) $\beta(g,h)\beta(h,g) = 1$ for all $g,h \in G$
(2) $\beta(g,h+k) = \beta(g,h)\beta(g,k)$ for all $g,h,k \in G$
(3) $\beta(g+h,k) = \beta(g,k)\beta(h,k)$ for all $g,h,k \in G$

The definition above implies, in particular, the following relations:

$$\beta(e,g) = \beta(g,e) = 1, \quad \beta(g,h) = \beta(-h,g), \quad \beta(g,g) = \pm 1 \quad \forall g,h \in G$$

where e denotes the identity element of G. In particular, fixing g one element of G, the induced mapping $\beta_g : G \longrightarrow \mathbb{F} \setminus \{0\}$ defines a homomorphism of groups.

Definition 2.3. Let G be an abelian group and β a commutation factor. The (complex or real) $G-$graded algebra

$$L = \bigoplus_{g\in G} Lg$$

with bracket product $[\ ,\ ]$, is called a (G,β)**-color Lie superalgebra** if for any $X \in L_g,\ Y \in L_h$, and $Z \in L$ we have

(1) $[X,Y] = -\beta(g,h)[Y,X]$ (anticommutative identity)
(2) $[[X,Y],Z] = [X,[Y,Z]] - \beta(g,h)[Y,[X,Z]]$ (Jacobi identity)

Remark 2.4. The Jacobi identity above can be rewritten in equivalent form as

$$\beta(k,g)[X,[Y,Z]] + \beta(h,k)[Z,[X,Y]] + \beta(g,h)[Y,[Z,X]] = 0$$

for all $X \in L_g$, $Y \in L_h$ and $Z \in L_k$.

Note that from the above definition we have the following consequences.

Corollary 2.4.1. *Let $L = \bigoplus_{g\in G} Lg$ be a (G,β)-color Lie superalgebra. Then we have*

(1) *L_e is a (complex or real) Lie algebra where e denotes the identity element of G.*
(2) *For all $g \in G \setminus \{e\}$, L_g is a representation of L_e. If $X \in L_e$ and $Y \in L_g$, then $[X,Y]$ denotes the action of X on Y.*

Examples. (1) For the particular case $G = \{e\}$, $L = L_e$ reduces to a Lie algebra.

(2) If $G = \mathbb{Z}_2 = \{0,1\}$ and $\beta(1,1) = -1$ we have *ordinary Lie superalgebras*, i.e. a *Lie superalgebra* is a $(\mathbb{Z}_2,\beta)$-color Lie superalgebra where $\beta(i,j) = (-1)^{ij}$ for all $i,j \in \mathbb{Z}_2$.

(3) As we always have $\beta(g,g) = \pm 1$, then we can set $G_+ = \{g \in G/\beta(g,g) = 1\}$ and $G_- = \{g \in G/\beta(g,g) = -1\}$. If $G = G_+$ such (G_+,β)-color Lie superalgebras are called color Lie algebras.

(4) If $A = \bigoplus_{g\in G} A_g$ is a G-graded associative algebra, then setting

$$[X,Y] = XY - \beta(g,h)YX$$

for $X \in \mathcal{A}_g,\ Y \in \mathcal{A}_h$, we make A into (G,β)-color Lie superalgebra $[A]_\beta$.

Definition 2.5. A **representation** of a (G,β)-color Lie superalgebra is a mapping $\rho : L \longrightarrow End(V)$, where $V = \bigoplus_{g\in G} V_g$ is a graded vector space such that

$$[\rho(X), \rho(Y)] = \rho(X)\rho(Y) - \beta(g,h)\rho(Y)\rho(X)$$

for all $X \in L_g,\ Y \in L_h$

We observe that for all $g, h \in G$ we have $\rho(L_g)V_h \subseteq V_{g+h}$, which implies that any V_g has the structure of a L_e-module. In particular considering the adjoint representation ad_L we have that every L_g has structure of L_e-module.

Two (G,β)-color Lie superalgebras L and M are called **isomorphic** if there is a linear isomorphism $\varphi : L \longrightarrow M$ such that $\varphi(L_g) = M_g$ for any $g \in G$ and also $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ for any $x, y \in L$.

Let $L = \bigoplus_{g\in G} Lg$ be a (G,β)-color Lie superalgebra. The **descending central sequence** of L is defined by

$$\mathcal{C}^0(L) = L, \quad \mathcal{C}^{k+1}(L) = [\mathcal{C}^k(L), L] \quad \forall k \geq 0$$

If $\mathcal{C}^k(L) = \{0\}$ for some k, the (G,β)-color Lie superalgebra is called **nilpotent**. The smallest integer k such as $\mathcal{C}^k(L) = \{0\}$ is called the **nilindex** of L.

Also, we are going to define some new **descending sequences of ideals**.

Definition 2.6. Let $L = \bigoplus_{g\in G} Lg$ be a (G,β)-color Lie superalgebra. Then, we define the new descending sequences of ideals $\mathcal{C}^k(L_e)$ (where e denotes the identity element of G) and $\mathcal{C}^k(L_g)$ with $g \in G \setminus \{e\}$, as follows:

$$\mathcal{C}^0(L_e) = L_e, \quad \mathcal{C}^{k+1}(L_e) = [L_e, \mathcal{C}^k(L_e)], \quad k \geq 0$$

and

$$\mathcal{C}^0(L_g) = L_g, \quad \mathcal{C}^{k+1}(L_g) = [L_e, \mathcal{C}^k(L_g)], \quad k \geq 0,\ g \in G \setminus \{e\}$$

Using the descending sequences of ideals defined above we give an invariant of color Lie superalgebras called **color-nilindex**. We are going to particularize this definition for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Definition 2.7. If $L = L_e \oplus L_a \oplus L_b \oplus L_c$ is a nilpotent $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta)$-color Lie superalgebra, then L has **color-nilindex** (p_e, p_a, p_b, p_c), if the following conditions holds:

$$(\mathcal{C}^{p_e-1}(L_e))(\mathcal{C}^{p_a-1}(L_a))(\mathcal{C}^{p_b-1}(L_b))(\mathcal{C}^{p_c-1}(L_c)) \neq 0$$

and

$$\mathcal{C}^{p_e}(Ł_e) = \mathcal{C}^{p_a}(Ł_a) = \mathcal{C}^{p_b}(Ł_b) = \mathcal{C}^{p_c}(Ł_c) = 0$$

Definition 2.8. Let $L = \bigoplus_{g\in G} Lg$ be a (G,β)-color Lie superalgebra. L_g is called a L_e**-filiform module** if there exists a decreasing subsequence of vectorial subspaces in its underlying vectorial space V, $V = V_m \supset \cdots \supset V_1 \supset V_0$, with dimensions $m, m-1, \ldots 0$, respectively, $m > 0$, and such that $[L_e, V_{i+1}] = V_i$.

Definition 2.9. Let $L = \bigoplus_{g\in G} Lg$ be a (G,β)-color Lie superalgebra. Then L is a **filiform color Lie superalgebra** if the following conditions hold:

(1) L_e is a filiform Lie algebra where e denotes the identity element of G.
(2) L_g has structure of L_e-filiform module, for all $g \in G \setminus \{e\}$

For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ we give another equivalent definition for filiform color Lie superalgebras using the invariant called color-nilindex.

Definition 2.10. Any $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta)$-color Lie superalgebra $L = L_e \oplus L_a \oplus L_b \oplus L_c$ is a **filiform color Lie superalgebra** if its color-nilindex is exactly

$$(dimL_e - 1, dimL_a, dimL_b, dimL_c)$$

It is not difficult to see that for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, there are, up to symmetries, four possible commutation factors β, i.e. β_1, β_2, β_3 and β_4 with:

1. $\beta_1(a,a) = \beta_1(b,b) = \beta_1(a,c) = \beta_1(b,c) = -1$
2. $\beta_2(a,a) = \beta_2(b,b) = \beta_2(a,b) = -1$
3. $\beta_3(a,b) = \beta_3(a,c) = \beta_3(b,c) = -1$

in all other cases $\beta_i(-,-) = 1$ with $i \in \{1,2,3\}$, thus $\beta_4 \equiv 1$.

Fixing a β_i $(1 \leq i \leq 4)$, we will note by $\mathcal{L}^{n,m,p,t}$ the variety of all $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebras $L = L_e \oplus L_a \oplus L_b \oplus L_c$ with $dim(L_e) = n+1$, $dim(L_a) = m$, $dim(L_b) = p$ and $dim(L_c) = t$.

Thus, $\mathcal{N}^{n,m,p,t}_{q,r,s,u}$ is the subset of $\mathcal{L}^{n,m,p,t}$ formed by all $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebras with color-nilindex (t_0, t_1, t_2, t_3) where $t_0 \leq q$, $t_1 \leq r$, $t_2 \leq s$ and $t_3 \leq u$. We observe that the set $\mathcal{N}^{n,m,p,t}_{n,m,p,t}$ is the variety of all nilpotent $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebras. For simplicity we write $\mathcal{N}^{n,m,p,t}$ instead of $\mathcal{N}^{n,m,p,t}_{n,m,p,t}$.

We denote by $\mathcal{F}^{n,m,p,t}$ the subset of $\mathcal{N}^{n,m,p,t}$ composed of all filiform color Lie superalgebras.

In the particular case of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ the theorem of adapted basis rests as follows for $L = L_e \oplus Ł_a \oplus L_b \oplus L_c \in \mathcal{F}^{n,m,p,t}$:

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1, \\ [X_0, X_n] = 0, & \\ [X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1, \\ [X_0, Y_m] = 0 & \\ [X_0, Z_k] = Z_{k+1}, & 1 \leq k \leq p-1, \\ [X_0, Z_p] = 0. & \\ [X_0, W_s] = W_{s+1}, & 1 \leq s \leq t-1, \\ [X_0, W_t] = 0. & \end{cases}$$

with $\{X_0, X_1, \ldots,\ X_n\}$ a basis of L_e, $\{Y_1, \ldots, Y_m\}$ a basis of L_a, $\{Z_1, \ldots, Z_p\}$ a basis of L_b and $\{W_1, \ldots, W_t\}$ a basis of L_c.

The model filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebra, $L^{n,m,p,t}$, is the simplest filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebra and it is defined in an adapted basis $\{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_p, W_1, \ldots, W_t\}$ by the following non-null bracket products

$$L^{n,m,p,t} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1 \\ [X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1 \\ [X_0, Z_k] = Z_{k+1} & 1 \leq k \leq p-1 \\ [X_0, W_s] = W_{s+1}, & 1 \leq s \leq t-1 \end{cases}$$

We observe that this definition does not depend on the election of the commutation factor β_i.

3. COCYCLES AND INFINITESIMAL DEFORMATIONS

Recall that a *module* $V = V_e \oplus V_a \oplus V_b \oplus V_c$ of the $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebra L is a bilinear map of degree $e = (\overline{0}, \overline{0})$, $L \times V \to V$ satisfying

$$\forall\ X \in L_g,\ Y \in L_h\ v \in V:\ X(Yv) - \beta_i(g,h)Y(Xv) = [X,Y]v$$

color Lie superalgebra cohomology is defined in the following well-known way (see e.g. [11]): in particular, the superspace of *q-dimensional cocycles* of the $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebra $L = L_e \oplus L_a \oplus L_b \oplus L_c$ with coefficients in the L-module $V_e \oplus V_a \oplus V_b \oplus V_c$ will be given, up to symmetry, by

- If $\beta_i = \beta_1$ or $\beta_i = \beta_2$, then
$$C^q(L;V) = \bigoplus_{q_e+q_a+q_b+q_c=q} \mathrm{Hom}\left(\wedge^{q_e} L_e \otimes S^{q_a} L_a \otimes S^{q_b} L_b \otimes \wedge^{q_c} L_c, V\right)$$
- If $\beta_i = \beta_3$ or $\beta_i = \beta_4$, then
$$C^q(L;V) = \bigoplus_{q_e+q_a+q_b+q_c=q} \mathrm{Hom}\left(\wedge^{q_e} L_e \otimes \wedge^{q_a} L_a \otimes \wedge^{q_b} L_b \otimes \wedge^{q_c} L_c, V\right)$$

This space is graded by $C^q(L;V) = C^q_e(L;V) \oplus C^q_a(L;V) \oplus C^q_b(L;V) \oplus C^q_c(L;V)$ with

- If $\beta_i = \beta_1$ or $\beta_i = \beta_2$
$$C^q_p(L;V) = \bigoplus_{\substack{q_e+q_a+q_b+q_c=q \\ q_b+q_c+p_1 \equiv r_1 \ mod\ 2 \\ q_a+q_c+p_2 \equiv r_2 \ mod\ 2}} \mathrm{Hom}\left(\wedge^{q_e} L_e \otimes S^{q_a} L_a \otimes S^{q_b} L_b \otimes \wedge^{q_c} L_c, V_r\right)$$

with $p = (p_1, p_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and $r = (r_1, r_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

- If $\beta_i = \beta_3$ or $\beta_i = \beta_4$
$$C^q_p(L;V) = \bigoplus_{\substack{q_e+q_a+q_b+q_c=q \\ q_b+q_c+p_1 \equiv r_1 \ mod\ 2 \\ q_a+q_c+p_2 \equiv r_2 \ mod\ 2}} \mathrm{Hom}\left(\wedge^{q_e} L_e \otimes \wedge^{q_a} L_a \otimes \wedge^{q_b} L_b \otimes \wedge^{q_c} L_c, V_r\right)$$

with $p = (p_1, p_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and $r = (r_1, r_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

The *coboundary operator* $\delta^q : C^q(L;V) \longrightarrow C^{q+1}(L;V)$, with $\delta^{q+1} \circ \delta^q = 0$ is defined in general, with L an arbitrary (G,β)-color Lie superalgebra and V an L-module, by the following formula for $q \geq 1$

$$
\begin{aligned}
(\delta^q g)\big(A_0, A_1, \dots, A_q\big) = \\
&\sum_{r=0}^{q} (-1)^r \beta(\gamma + \alpha_0 + \dots + \alpha_{r-1}, \alpha_r) A_r \cdot g\big(A_0, \dots, \hat{A}_r, \dots, A_q\big) \\
+ \sum_{r<s} (-1)^s \beta(\alpha_{r+1} + \dots + \alpha_{s-1}, \alpha_s) g\big(A_0, \dots, A_{r-1}, [A_r, A_s], A_{r+1}, \dots, \hat{A}_s, \dots A_q\big),
\end{aligned}
$$

where $g \in C^q(L;V)$ of degree γ, and $A_0, A_1, \dots, A_q \in L$ are homogeneous with degrees $\alpha_0, \alpha_1, \dots, \alpha_q$ respectively. The sign $\hat{}$ indicates that the element below it must be omitted and empty sums (like $\alpha_0 + \dots + \alpha_{r-1}$ for $r = 0$ and $\alpha_{r+1} + \dots + \alpha_{s-1}$ for $s = r + 1$) are set equal to zero. In particular, for $q = 1$ we have

$$
(\delta^1 g)(A_0, A_1) = \beta(\gamma, \alpha_0) A_0 \cdot g(A_1) - \beta(\gamma + \alpha_0, \alpha_1) A_1 \cdot g(A_0) - g([A_0, A_1])
$$

and for $q = 2$ we obtain

$$
\begin{aligned}
(\delta^2 g)\big(A_0, A_1, A_2\big) \quad = \quad & \beta(\gamma, \alpha_0) A_0 \cdot g\big(A_1, A_2\big) - \beta(\gamma + \alpha_0, \alpha_1) A_1 \cdot g\big(A_0, A_2\big) + \\
& \beta(\gamma + \alpha_0 + \alpha_1, \alpha_2) A_2 \cdot g\big(A_0, A_1\big) \\
& - g([A_0, A_1], A_2) + \beta(\alpha_1, \alpha_2) g([A_0, A_2], A_1) + g(A_0, [A_1, A_2]).
\end{aligned}
$$

Let $Z^q(L;V)$ denote the kernel of δ^q and let $B^q(L;V)$ denote the image of δ^{q-1}, then we have that $B^q(L;V) \subset Z^q(L;V)$. The elements of $Z^q(L;V)$ are called *q-cocycles*, the elements of $B^q(L;V)$ are the *q-coboundaries*. Thus, we can constuct the so-called *cohomology groups*

$$
H^q(L;V) = Z^q(L;V) \big/ B^q(L;V)
$$

$$
H^q_p(L;V) = Z^q_p(L;V) \big/ B^q_p(L;V) \text{ , } if \ G = \mathbb{Z}_2 \times \mathbb{Z}_2 \ then \ p = e, a, b, c
$$

Two elements of $Z^q(L;V)$ are said to be *cohomologous* if their residue classes modulo $B^q(L;V)$ coincide, i.e., if their difference lies in $B^q(L;V)$.

We will focus our study in the 2-cocycles of degree $e = (\overline{0}, \overline{0})$, $Z^2_e(L^{n,m,p,t}; L^{n,m,p,t})$ with $L^{n,m,p,t}$ the model filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebra. Taking into account the law of $L^{n,m,p,t}$ the condition that have to verify $\psi \in C^2_e(L^{n,m,p,t}; L^{n,m,p,t})$ to be a 2-cocycle rests

$$
\begin{aligned}
(\delta^2 \psi)\big(A_0, A_1, A_2\big) \quad = \quad & [A_0, \psi\big(A_1, A_2\big)] - \beta_i(\alpha_0, \alpha_1)[A_1, \psi\big(A_0, A_2\big)] + \\
& \beta_i(\alpha_0 + \alpha_1, \alpha_2)[A_2, \psi\big(A_0, A_1\big)] - \psi([A_0, A_1], A_2) \\
& + \beta_i(\alpha_1, \alpha_2) \psi([A_0, A_2], A_1) + \psi(A_0, [A_1, A_2]) = 0
\end{aligned}
$$

for all $A_0, A_1, A_2 \in L^{n,m,p,t}$. We observe that $L^{n,m,p,t}$ has structure of $L^{n,m,p,t}$-module via the adjoint representation.

We consider an homogeneous basis of $L^{n,m,p,t} = L_e \oplus L_a \oplus L_b \oplus L_c$, in particular an adapted basis $\{X_0, X_1, \dots, X_n, Y_1, \dots, Y_m, Z_1, \dots, Z_p, W_1, \dots, W_t\}$ with $\{X_0, X_1, \dots, X_n\}$ a basis of L_e, $\{Y_1, \dots, Y_m\}$ a basis of L_a, $\{Z_1, \dots, Z_p\}$ a basis of L_b and $\{W_1, \dots, W_t\}$ a basis of L_c.

Under these conditions we have the following lemma.

Lemma 3.1. *Let* ψ *be such that* $\psi \in C_e^2(L^{n,m,p,t}; L^{n,m,p,t})$*, then* ψ *is a 2-cocycle,* $\psi \in Z_e^2(L^{n,m,p,t}; L^{n,m,p,t})$*, iff the below* 20 *conditions hold for all* $X_i, X_j, X_k \in L_e$*,* $Y_i, Y_j, Y_k \in L_a$*,* $Z_i, Z_j, Z_k \in L_b$ *and* $W_i, W_j, W_k \in L_c$*.*

(1) $[X_i, \psi(X_j, X_k)] - [X_j, \psi(X_i, X_k)] + [X_k, \psi(X_i, X_j)] - \psi([X_i, X_j], X_k) + \psi([X_i, X_k], X_j) + \psi(X_i, [X_j, X_k]) = 0$

(2) $[X_i, \psi(X_j, Y_k)] - [X_j, \psi(X_i, Y_k)] + [Y_k, \psi(X_i, X_j)] - \psi([X_i, X_j], Y_k) + \psi([X_i, Y_k], X_j) + \psi(X_i, [X_j, Y_k]) = 0$

(3) $[X_i, \psi(X_j, Z_k)] - [X_j, \psi(X_i, Z_k)] + [Z_k, \psi(X_i, X_j)] - \psi([X_i, X_j], Z_k) + \psi([X_i, Z_k], X_j) + \psi(X_i, [X_j, Z_k]) = 0$

(4) $[X_i, \psi(X_j, W_k)] - [X_j, \psi(X_i, W_k)] + [W_k, \psi(X_i, X_j)] - \psi([X_i, X_j], W_k) + \psi([X_i, W_k], X_j) + \psi(X_i, [X_j, W_k]) = 0$

(5.1) $\beta_i = \beta_1$ *or* $\beta_i = \beta_2$
$[X_i, \psi(Y_j, Y_k)] - [Y_j, \psi(X_i, Y_k)] - [Y_k, \psi(X_i, Y_j)] - \psi([X_i, Y_j], Y_k) - \psi([X_i, Y_k], Y_j) + \psi(X_i, [Y_j, Y_k]) = 0$

(5.2) $\beta_i = \beta_3$ *or* $\beta_i = \beta_4$
$[X_i, \psi(Y_j, Y_k)] - [Y_j, \psi(X_i, Y_k)] + [Y_k, \psi(X_i, Y_j)] - \psi([X_i, Y_j], Y_k) + \psi([X_i, Y_k], Y_j) + \psi(X_i, [Y_j, Y_k]) = 0$

(6.1) $\beta_i = \beta_1$ *or* $\beta_i = \beta_4$
$[X_i, \psi(Y_j, Z_k)] - [Y_j, \psi(X_i, Z_k)] + [Z_k, \psi(X_i, Y_j)] - \psi([X_i, Y_j], Z_k) + \psi([X_i, Z_k], Y_j) + \psi(X_i, [Y_j, Z_k]) = 0$

(6.2) $\beta_i = \beta_2$ *or* $\beta_i = \beta_3$
$[X_i, \psi(Y_j, Z_k)] - [Y_j, \psi(X_i, Z_k)] - [Z_k, \psi(X_i, Y_j)] - \psi([X_i, Y_j], Z_k) - \psi([X_i, Z_k], Y_j) + \psi(X_i, [Y_j, Z_k]) = 0$

(7.1) $\beta_i = \beta_1$ *or* $\beta_i = \beta_3$
$[X_i, \psi(Y_j, W_k)] - [Y_j, \psi(X_i, W_k)] - [W_k, \psi(X_i, Y_j)] - \psi([X_i, Y_j], W_k) - \psi([X_i, W_k], Y_j) + \psi(X_i, [Y_j, W_k]) = 0$

(7.2) $\beta_i = \beta_2$ *or* $\beta_i = \beta_4$
$[X_i, \psi(Y_j, W_k)] - [Y_j, \psi(X_i, W_k)] + [W_k, \psi(X_i, Y_j)] - \psi([X_i, Y_j], W_k) + \psi([X_i, W_k], Y_j) + \psi(X_i, [Y_j, W_k]) = 0$

(8.1) $\beta_i = \beta_1$ *or* $\beta_i = \beta_2$
$[X_i, \psi(Z_j, Z_k)] - [Z_j, \psi(X_i, Z_k)] - [Z_k, \psi(X_i, Z_j)] - \psi([X_i, Z_j], Z_k) - \psi([X_i, Z_k], Z_j) + \psi(X_i, [Z_j, Z_k]) = 0$

(8.2) $\beta_i = \beta_3$ *or* $\beta_i = \beta_4$
$[X_i, \psi(Z_j, Z_k)] - [Z_j, \psi(X_i, Z_k)] + [Z_k, \psi(X_i, Z_j)] - \psi([X_i, Z_j], Z_k) + \psi([X_i, Z_k], Z_j) + \psi(X_i, [Z_j, Z_k]) = 0$

(9.1) $\beta_i = \beta_1$ *or* $\beta_i = \beta_3$
$[X_i, \psi(Z_j, W_k)] - [Z_j, \psi(X_i, W_k)] - [W_k, \psi(X_i, Z_j)] - \psi([X_i, Z_j], W_k) -$
$\psi([X_i, W_k], Z_j) + \psi(X_i, [Z_j, W_k]) = 0$

(9.2) $\beta_i = \beta_2$ *or* $\beta_i = \beta_4$
$[X_i, \psi(Z_j, W_k)] - [Z_j, \psi(X_i, W_k)] + [W_k, \psi(X_i, Z_j)] - \psi([X_i, Z_j], W_k) +$
$\psi([X_i, W_k], Z_j) + \psi(X_i, [Z_j, W_k]) = 0$

(10) $[X_i, \psi(W_j, W_k)] - [W_j, \psi(X_i, W_k)] + [W_k, \psi(X_i, W_j)] - \psi([X_i, W_j], W_k) +$
$\psi([X_i, W_k], W_j) + \psi(X_i, [W_j, W_k]) = 0$

(11.1) $\beta_i = \beta_1$ *or* $\beta_i = \beta_2$
$[Y_i, \psi(Y_j, Y_k)] + [Y_j, \psi(Y_i, Y_k)] + [Y_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], Y_k) -$
$\psi([Y_i, Y_k], Y_j) + \psi(Y_i, [Y_j, Y_k]) = 0$

(11.2) $\beta_i = \beta_3$ *or* $\beta_i = \beta_4$
$[Y_i, \psi(Y_j, Y_k)] - [Y_j, \psi(Y_i, Y_k)] + [Y_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], Y_k) +$
$\psi([Y_i, Y_k], Y_j) + \psi(Y_i, [Y_j, Y_k]) = 0$

(12.1) $\beta_i = \beta_1$
$[Y_i, \psi(Y_j, Z_k)] + [Y_j, \psi(Y_i, Z_k)] + [Z_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], Z_k) +$
$\psi([Y_i, Z_k], Y_j) + \psi(Y_i, [Y_j, Z_k]) = 0$

(12.2) $\beta_i = \beta_2$
$[Y_i, \psi(Y_j, Z_k)] + [Y_j, \psi(Y_i, Z_k)] + [Z_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], Z_k) -$
$\psi([Y_i, Z_k], Y_j) + \psi(Y_i, [Y_j, Z_k]) = 0$

(12.3) $\beta_i = \beta_3$
$[Y_i, \psi(Y_j, Z_k)] - [Y_j, \psi(Y_i, Z_k)] + [Z_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], Z_k) -$
$\psi([Y_i, Z_k], Y_j) + \psi(Y_i, [Y_j, Z_k]) = 0$

(12.4) $\beta_i = \beta_4$
$[Y_i, \psi(Y_j, Z_k)] - [Y_j, \psi(Y_i, Z_k)] + [Z_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], Z_k) +$
$\psi([Y_i, Z_k], Y_j) + \psi(Y_i, [Y_j, Z_k]) = 0$

(13.1) $\beta_i = \beta_1$
$[Y_i, \psi(Y_j, W_k)] + [Y_j, \psi(Y_i, W_k)] + [W_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], W_k) -$
$\psi([Y_i, W_k], Y_j) + \psi(Y_i, [Y_j, W_k]) = 0$

(13.2) $\beta_i = \beta_2$
$[Y_i, \psi(Y_j, W_k)] + [Y_j, \psi(Y_i, W_k)] + [W_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], W_k) +$
$\psi([Y_i, W_k], Y_j) + \psi(Y_i, [Y_j, W_k]) = 0$

(13.3) $\beta_i = \beta_3$
$[Y_i, \psi(Y_j, W_k)] - [Y_j, \psi(Y_i, W_k)] + [W_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], W_k) -$
$\psi([Y_i, W_k], Y_j) + \psi(Y_i, [Y_j, W_k]) = 0$

(13.4) $\beta_i = \beta_4$
$[Y_i, \psi(Y_j, W_k)] - [Y_j, \psi(Y_i, W_k)] + [W_k, \psi(Y_i, Y_j)] - \psi([Y_i, Y_j], W_k) +$
$\psi([Y_i, W_k], Y_j) + \psi(Y_i, [Y_j, W_k]) = 0$

(14.1) $\beta_i = \beta_1$
$[Y_i, \psi(Z_j, Z_k)] - [Z_j, \psi(Y_i, Z_k)] - [Z_k, \psi(Y_i, Z_j)] - \psi([Y_i, Z_j], Z_k) -$
$\psi([Y_i, Z_k], Z_j) + \psi(Y_i, [Z_j, Z_k]) = 0$

(14.2) $\beta_i = \beta_2$
$[Y_i, \psi(Z_j, Z_k)] + [Z_j, \psi(Y_i, Z_k)] + [Z_k, \psi(Y_i, Z_j)] - \psi([Y_i, Z_j], Z_k) -$
$\psi([Y_i, Z_k], Z_j) + \psi(Y_i, [Z_j, Z_k]) = 0$

(14.3) $\beta_i = \beta_3$
$[Y_i, \psi(Z_j, Z_k)] + [Z_j, \psi(Y_i, Z_k)] - [Z_k, \psi(Y_i, Z_j)] - \psi([Y_i, Z_j], Z_k) +$
$\psi([Y_i, Z_k], Z_j) + \psi(Y_i, [Z_j, Z_k]) = 0$

(14.4) $\beta_i = \beta_4$
$[Y_i, \psi(Z_j, Z_k)] - [Z_j, \psi(Y_i, Z_k)] + [Z_k, \psi(Y_i, Z_j)] - \psi([Y_i, Z_j], Z_k) +$
$\psi([Y_i, Z_k], Z_j) + \psi(Y_i, [Z_j, Z_k]) = 0$

(15.1) $\beta_i = \beta_1$
$[Y_i, \psi(Z_j, W_k)] - [Z_j, \psi(Y_i, W_k)] + [W_k, \psi(Y_i, Z_j)] - \psi([Y_i, Z_j], W_k) -$
$\psi([Y_i, W_k], Z_j) + \psi(Y_i, [Z_j, W_k]) = 0$

(15.2) $\beta_i = \beta_2$
$[Y_i, \psi(Z_j, W_k)] + [Z_j, \psi(Y_i, W_k)] + [W_k, \psi(Y_i, Z_j)] - \psi([Y_i, Z_j], W_k) +$
$\psi([Y_i, W_k], Z_j) + \psi(Y_i, [Z_j, W_k]) = 0$

(15.3) $\beta_i = \beta_3$
$[Y_i, \psi(Z_j, W_k)] + [Z_j, \psi(Y_i, W_k)] + [W_k, \psi(Y_i, Z_j)] - \psi([Y_i, Z_j], W_k) -$
$\psi([Y_i, W_k], Z_j) + \psi(Y_i, [Z_j, W_k]) = 0$

(15.4) $\beta_i = \beta_4$
$[Y_i, \psi(Z_j, W_k)] - [Z_j, \psi(Y_i, W_k)] + [W_k, \psi(Y_i, Z_j)] - \psi([Y_i, Z_j], W_k) +$
$\psi([Y_i, W_k], Z_j) + \psi(Y_i, [Z_j, W_k]) = 0$

(16.1) $\beta_i = \beta_1$ *or* $\beta_i = \beta_3$
$[Y_i, \psi(W_j, W_k)] + [W_j, \psi(Y_i, W_k)] - [W_k, \psi(Y_i, W_j)] - \psi([Y_i, W_j], W_k) +$
$\psi([Y_i, W_k], W_j) + \psi(Y_i, [W_j, W_k]) = 0$

(16.2) $\beta_i = \beta_2$ *or* $\beta_i = \beta_4$
$[Y_i, \psi(W_j, W_k)] - [W_j, \psi(Y_i, W_k)] + [W_k, \psi(Y_i, W_j)] - \psi([Y_i, W_j], W_k) +$
$\psi([Y_i, W_k], W_j) + \psi(Y_i, [W_j, W_k]) = 0$

(17.1) $\beta_i = \beta_1$ *or* $\beta_i = \beta_2$
$[Z_i, \psi(Z_j, Z_k)] + [Z_j, \psi(Z_i, Z_k)] + [Z_k, \psi(Z_i, Z_j)] - \psi([Z_i, Z_j], Z_k) -$
$\psi([Z_i, Z_k], Z_j) + \psi(Z_i, [Z_j, Z_k]) = 0$

(17.2) $\beta_i = \beta_3$ *or* $\beta_i = \beta_4$
$[Z_i, \psi(Z_j, Z_k)] - [Z_j, \psi(Z_i, Z_k)] + [Z_k, \psi(Z_i, Z_j)] - \psi([Z_i, Z_j], Z_k)+$
$\psi([Z_i, Z_k], Z_j) + \psi(Z_i, [Z_j, Z_k]) = 0$

(18.1) $\beta_i = \beta_1$
$[Z_i, \psi(Z_j, W_k)] + [Z_j, \psi(Z_i, W_k)] + [W_k, \psi(Z_i, Z_j)] - \psi([Z_i, Z_j], W_k)-$
$\psi([Z_i, W_k], Z_j) + \psi(Z_i, [Z_j, W_k]) = 0$

(18.2) $\beta_i = \beta_2$
$[Z_i, \psi(Z_j, W_k)] + [Z_j, \psi(Z_i, W_k)] + [W_k, \psi(Z_i, Z_j)] - \psi([Z_i, Z_j], W_k)+$
$\psi([Z_i, W_k], Z_j) + \psi(Z_i, [Z_j, W_k]) = 0$

(18.3) $\beta_i = \beta_3$
$[Z_i, \psi(Z_j, W_k)] - [Z_j, \psi(Z_i, W_k)] + [W_k, \psi(Z_i, Z_j)] - \psi([Z_i, Z_j], W_k)-$
$\psi([Z_i, W_k], Z_j) + \psi(Z_i, [Z_j, W_k]) = 0$

(18.4) $\beta_i = \beta_4$
$[Z_i, \psi(Z_j, W_k)] - [Z_j, \psi(Z_i, W_k)] + [W_k, \psi(Z_i, Z_j)] - \psi([Z_i, Z_j], W_k)+$
$\psi([Z_i, W_k], Z_j) + \psi(Z_i, [Z_j, W_k]) = 0$

(19.1) $\beta_i = \beta_1$ *or* $\beta_i = \beta_3$
$[Z_i, \psi(W_j, W_k)] + [W_j, \psi(Z_i, W_k)] - [W_k, \psi(Z_i, W_j)] - \psi([Z_i, W_j], W_k)+$
$\psi([Z_i, W_k], W_j) + \psi(Z_i, [W_j, W_k]) = 0$

(19.2) $\beta_i = \beta_2$ *or* $\beta_i = \beta_4$
$[Z_i, \psi(W_j, W_k)] - [W_j, \psi(Z_i, W_k)] + [W_k, \psi(Z_i, W_j)] - \psi([Z_i, W_j], W_k)+$
$\psi([Z_i, W_k], W_j) + \psi(Z_i, [W_j, W_k]) = 0$

(20) $[W_i, \psi(W_j, W_k)] - [W_j, \psi(W_i, W_k)] + [W_k, \psi(W_i, W_j)] - \psi([W_i, W_j], W_k)+$
$\psi([W_i, W_k], W_j) + \psi(W_i, [W_j, W_k]) = 0$

On the other hand, recall that an *infinitesimal deformation* φ of the $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebra L, $L = L_e \oplus L_a \oplus L_b \oplus L_c$, it is a bilinear map $\varphi : L \times L \longrightarrow L$ that satisfies the following two relations (for more details in general see [10]):

(1) $\varphi(X, Y) = -\beta_i(g, h)\varphi(Y, X)$, for all $X \in L_g$ and $Y \in L_h$.
(2) $\mu \circ \varphi + \varphi \circ \mu = 0$, with μ representing the law of L, i.e. for all $X \in L_g$, $Y \in L_h$ and $Z \in L_k$ we have

$$(\mu \circ \varphi + \varphi \circ \mu)(X, Y, Z) =$$
$$\beta_i(k, g)\mu(X, \varphi(Y, Z)) + \beta_i(h, k)\mu(Z, \varphi(X, Y)) + \beta_i(g, h)\mu(Y, \varphi(Z, X))+$$
$$\beta_i(k, g)\varphi(X, \mu(Y, Z)) + \beta_i(h, k)\varphi(Z, \mu(X, Y)) + \beta_i(g, h)\varphi(Y, \mu(Z, X) = 0$$

For an arbitrary group grading G and an admissible commutator factor β we have the following result

Theorem 3.1.1. [10] *(1) Any filiform (G,β)-color Lie superalgebra law μ is isomorphic to $\mu_0+\varphi$ where μ_0 is the law of the model filiform (G,β)-color Lie superalgebra and φ is an infinitesimal deformation of μ_0 verifying that $\varphi(X_0,X)=0$ for all $X\in L$, with X_0 the characteristic vector of the model one.*

(2) Conversely, if φ is an infinitesimal deformation of a model filiform (G,β)-color Lie superalgebra law μ_0 with $\varphi(X_0,X)=0$ for all $X\in L$, then the law $\mu_0+\varphi$ is a filiform (G,β)-color Lie superalgebra law iff $\varphi\circ\varphi=0$.

$\varphi\circ\varphi(X,Y,Z)=\beta(k,g)\varphi(X,\varphi(Y,Z))+\beta(h,k)\varphi(Z,\varphi(X,Y))+\beta(g,h)\varphi(Y,\varphi(Z,X)$ *for all $X\in L_g$, $Y\in L_h$ and $Z\in L_k$.*

Thus, any filiform $(\mathbb{Z}_2\times\mathbb{Z}_2,\beta_i)$-color Lie superalgebra will be a linear deformation of the model filiform $(\mathbb{Z}_2\times\mathbb{Z}_2,\beta_i)$-color Lie superalgebra, i.e. $L^{n,m,p,t}$ is the model filiform $(\mathbb{Z}_2\times\mathbb{Z}_2,\beta_i)$-color Lie superalgebra and another arbitrary filiform $(\mathbb{Z}_2\times\mathbb{Z}_2,\beta_i)$-color Lie superalgebra will be equal to $L^{n,m,p,t}+\varphi$, with φ an infinitesimal deformation of $L^{n,m,p,t}$. Hence the importance of these deformations.

Next, we present the correspondence between 2-cocycles and infinitesimal deformations.

Proposition 3.2. *ψ is an infinitesimal deformation of $L^{n,m,p,t}$ iff ψ is a 2-cocycle of degree e, $\psi\in Z_e^2(L^{n,m,p,t};L^{n,m,p,t})$.*

Proof. If ψ is an infinitesimal deformation of $L^{n,m,p,t}$, then we have for all $X\in L_g$, $Y\in L_h$ and $Z\in L_k$

(1) $\psi(X,Y)=-\beta_i(g,h)\psi(Y,X)$, and

(2) $\beta_i(k,g)[X,\psi(Y,Z)]+\beta_i(h,k)[Z,\psi(X,Y)]+\beta_i(g,h)[Y,\psi(Z,X)]+$
$\beta_i(k,g)\psi(X,[Y,Z])+\beta_i(h,k)\psi(Z,[X,Y])+\beta_i(g,h)\psi(Y,[Z,X])=0$

On the other hand the 2-cocycles $Z_e^2(L^{n,m,p,t};L^{n,m,p,t})$ are in particular β_i-skew symmetric which is equivalent to (1), and the condition to be a 2-cocycle

$$[A_0,\psi\big(A_1,A_2\big)]-\beta_i(\alpha_0,\alpha_1)[A_1,\psi\big(A_0,A_2\big)]+\beta_i(\alpha_0+\alpha_1,\alpha_2)[A_2,\psi\big(A_0,A_1\big)]$$
$$-\psi([A_0,A_1],A_2)+\beta_i(\alpha_1,\alpha_2)\psi([A_0,A_2],A_1)+\psi(A_0,[A_1,A_2])=0$$

is equivalent to (2) taking into account the different ways to express the identity of Jacobi. □

So, in order to determine all the filiform $(\mathbb{Z}_2\times\mathbb{Z}_2,\beta_i)$-color Lie superalgebras it is only necessary to compute the infinitesimal deformations or so called 2-cocycles of degree e, that vanish on the characteristic vector X_0. Thanks to the following lemma these infinitesimal deformations will can be decomposed into 10 subspaces.

Note that for $G=\mathbb{Z}_2\times\mathbb{Z}_2$, there are, up to symmetries, four possible commutation factors β, i.e. β_1, β_2, β_3 and β_4:

1. $\beta_1(a,a)=\beta_1(b,b)=\beta_1(a,c)=\beta_1(b,c)=-1$
2. $\beta_2(a,a)=\beta_2(b,b)=\beta_2(a,b)=-1$
3. $\beta_3(a,b)=\beta_3(a,c)=\beta_3(b,c)=-1$

in all other cases $\beta_i(-,-)=1$ with $i\in\{1,2,3\}$, thus $\beta_4\equiv 1$.

Thus, the decomposition into 10 subspaces of the 2-cocycles of degree e depends on the election of β_i. This leads to the following lemma.

Lemma 3.3. *Let $Z^2(L;L)$ be the 2-cocycles $Z^2_e(L^{n,m,p,t};L^{n,m,p,t})$ that vanish on the characteristic vector X_0 with $L^{n,m,p,t} = L = L_e \oplus L_a \oplus L_b \oplus L_c$. Then, $Z^2(L;L)$ can be divided into ten subspaces, i.e.*

1. *If $\beta_i = \beta_1$, then*

$$\begin{aligned} Z^2(L;L) = \quad & Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_e, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_a, L_a) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_b, L_b) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_c, L_c) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(S^2 L_a, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(S^2 L_b, L_e) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_c \wedge L_c, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_a \wedge L_b, L_c) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_a \otimes L_c, L_b) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_b \otimes L_c, L_a) \\ = \quad & H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5 \oplus H_6 \oplus H_7 \oplus H_8 \oplus H_9 \oplus H_{10} \end{aligned}$$

2. *If $\beta_i = \beta_2$, then*

$$\begin{aligned} Z^2(L;L) = \quad & Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_e, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_a, L_a) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_b, L_b) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_c, L_c) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(S^2 L_a, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(S^2 L_b, L_e) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_c \wedge L_c, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_a \otimes L_b, L_c) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_a \wedge L_c, L_b) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_b \wedge L_c, L_a) \\ = \quad & H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5 \oplus H_6 \oplus H_7 \oplus H_8 \oplus H_9 \oplus H_{10} \end{aligned}$$

3. *If $\beta_i = \beta_3$, then*

$$\begin{aligned} Z^2(L;L) = \quad & Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_e, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_a, L_a) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_b, L_b) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_c, L_c) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_a \wedge L_a, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_b \wedge L_b, L_e) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_c \wedge L_c, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_a \otimes L_b, L_c) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_a \otimes L_c, L_b) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_b \otimes L_c, L_a) \\ = \quad & H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5 \oplus H_6 \oplus H_7 \oplus H_8 \oplus H_9 \oplus H_{10} \end{aligned}$$

4. *If $\beta_i = \beta_4$, then*

$$\begin{aligned} Z^2(L;L) = \quad & Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_e, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_a, L_a) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_b, L_b) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_e \wedge L_c, L_c) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_a \wedge L_a, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_b \wedge L_b, L_e) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_c \wedge L_c, L_e) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_a \wedge L_b, L_c) \oplus \\ & Z^2(L;L) \cap \mathrm{Hom}(L_a \wedge L_c, L_b) \oplus Z^2(L;L) \cap \mathrm{Hom}(L_b \wedge L_c, L_a) \\ = \quad & H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5 \oplus H_6 \oplus H_7 \oplus H_8 \oplus H_9 \oplus H_{10} \end{aligned}$$

Proof. Case 1. $\beta_i = \beta_1$, let φ be such that $\varphi \in Z^2_e(L^{n,m,p,t};L^{n,m,p,t})$ and $\varphi(X_0, X) = 0 \ \forall X \in L$. It is easy to see that φ can be decomposed as $\varphi = h_1 + h_2 + h_3 + h_4 + h_5 + h_6 + h_7 + h_8 + h_9 + h_{10}$ with $h_1 \in \mathrm{Hom}(L_e \wedge L_e, L_e), h_2 \in \mathrm{Hom}(L_e \wedge L_a, L_a), h_3 \in \mathrm{Hom}(L_e \wedge L_b, L_b), h_4 \in \mathrm{Hom}(L_e \wedge L_c, L_c), h_5 \in \mathrm{Hom}(S^2 L_a, L_e), h_6 \in \mathrm{Hom}(S^2 L_b, L_e), h_7 \in \mathrm{Hom}(L_c \wedge L_c, L_e), h_8 \in \mathrm{Hom}(L_a \wedge L_b, L_c), h_9 \in \mathrm{Hom}(L_a \otimes L_c, L_b)$ and $h_{10} \in \mathrm{Hom}(L_b \otimes L_c, L_a)$.

To complete the proof it only remains to verify that each of the above homomorphisms is also a 2-cocycle.

As $\varphi = h_1 + h_2 + h_3 + h_4 + h_5 + h_6 + h_7 + h_8 + h_9 + h_{10}$ is a 2-cocycle it will verify the 20 equations of the Lemma 3.1, that rest

(1) $[X_i, h_1(X_j, X_k)] - [X_j, h_1(X_i, X_k)] + [X_k, h_1(X_i, X_j)] - h_1([X_i, X_j], X_k) + h_1([X_i, X_k], X_j) + h_1(X_i, [X_j, X_k]) = 0$

(2) $[X_i, h_2(X_j, Y_k)] - [X_j, h_2(X_i, Y_k)] + [Y_k, h_1(X_i, X_j)] - h_2([X_i, X_j], Y_k) + h_2([X_i, Y_k], X_j) + h_2(X_i, [X_j, Y_k]) = 0$

(3) $[X_i, h_3(X_j, Z_k)] - [X_j, h_3(X_i, Z_k)] + [Z_k, h_1(X_i, X_j)] - h_3([X_i, X_j], Z_k) + h_3([X_i, Z_k], X_j) + h_3(X_i, [X_j, Z_k]) = 0$

(4) $[X_i, h_4(X_j, W_k)] - [X_j, h_4(X_i, W_k)] + [W_k, h_1(X_i, X_j)] - h_4([X_i, X_j], W_k) + h_4([X_i, W_k], X_j) + h_4(X_i, [X_j, W_k]) = 0$

(5) $[X_i, h_5(Y_j, Y_k)] - h_5([X_i, Y_j], Y_k) - h_5([X_i, Y_k], Y_j) = 0$

(6) $[X_i, h_8(Y_j, Z_k)] - h_8([X_i, Y_j], Z_k) + h_8([X_i, Z_k], Y_j) = 0$

(7) $[X_i, h_9(Y_j, W_k)] - h_9([X_i, Y_j], W_k) - h_9([X_i, W_k], Y_j) = 0$

(8) $[X_i, h_6(Z_j, Z_k)]] - h_6([X_i, Z_j], Z_k) - h_6([X_i, Z_k], Z_j) = 0$

(9) $[X_i, h_{10}(Z_j, W_k)] - h_{10}([X_i, Z_j], W_k) - h_{10}([X_i, W_k], Z_j) = 0$

(10) $[X_i, h_7(W_j, W_k)] - h_7([X_i, W_j], W_k) + h_7([X_i, W_k], W_j) = 0$

(11) $[Y_i, h_5(Y_j, Y_k)] - [Y_j, h_5(Y_i, Y_k)] + [Y_k, h_5(Y_i, Y_j)] = 0$

(12) $[Z_k, h_5(Y_i, Y_j)] = 0$

(13) $[W_k, h_5(Y_i, Y_j)] = 0$

(14) $[Y_i, h_6(Z_j, Z_k)] = 0$

(16) $[Y_i, h_7(W_j, W_k)]] = 0$

(17) $[Z_i, h_6(Z_j, Z_k)] + [Z_j, h_6(Z_i, Z_k)] + [Z_k, h_6(Z_i, Z_j)] = 0$

(18) $[W_k, h_6(Z_i, Z_j)] = 0$

(19) $[Z_i, h_7(W_j, W_k)] = 0$

(20) $[W_i, h_7(W_j, W_k)] - [W_j, h_7(W_i, W_k)] + [W_k, h_7(W_i, W_j)] = 0$

for all $X_i, X_j, X_k \in L_e$, $Y_i, Y_j, Y_k \in L_a$, $Z_i, Z_j, Z_k \in L_b$ and $W_i, W_j, W_k \in L_c$. From the equations (12) and (13) we obtain that $X_0 \notin \operatorname{Im} h_5$ and thus, the equation (11) vanishes. From the equations (14) and (18) we obtain that $X_0 \notin \operatorname{Im} h_6$ and thus, the equation (17) vanishes. Analogously from the equations (16) and (19) it can be obtained that $X_0 \notin \operatorname{Im} h_7$, and then the equation (20) vanishes. Finally,

from the equation (2) it can be obtained that $X_0 \notin \text{Im } h_1$, and so $X_0 \notin \text{Im } \varphi$. Thus, the equations (2), (3) and (4) can be rewritten as follows

(2) $[X_i, h_2(X_j, Y_k)] - [X_j, h_2(X_i, Y_k)] - h_2([X_i, X_j], Y_k) +$
$h_2([X_i, Y_k], X_j) + h_2(X_i, [X_j, Y_k]) = 0$

(3) $[X_i, h_3(X_j, Z_k)] - [X_j, h_3(X_i, Z_k)] - h_3([X_i, X_j], Z_k) +$
$h_3([X_i, Z_k], X_j) + h_3(X_i, [X_j, Z_k]) = 0$

(4) $[X_i, h_4(X_j, W_k)] - [X_j, h_4(X_i, W_k)] - h_4([X_i, X_j], W_k) +$
$h_4([X_i, W_k], X_j) + h_4(X_i, [X_j, W_k]) = 0$

each of the remaining equations, i.e. from (1) to (10), corresponds to the cocycle condition for a homomorphism. Analogously, it can be proved the cases 2, 3 and 4. □

If we fix, for example, $\beta_i = \beta_1$ then we can note that the infinitesimal deformations belonging to $H_5 = Z^2(L; L) \cap \text{Hom}(S^2 L_a, L_e)$ are all linearly integrable. In fact, let φ be such that $\varphi \in Z^2(L; L) \cap \text{Hom}(S^2 L_a, L_e)$, then $\varphi \circ \varphi$ rests

$$\varphi\circ\varphi(X, Y, Z) = \beta_1(k, g)\varphi(X, \varphi(Y, Z)) + \beta_1(h, k)\varphi(Z, \varphi(X, Y)) + \beta_1(g, h)\varphi(Y, \varphi(Z, X)$$

for all $X \in L_g$, $Y \in L_h$ and $Z \in L_k$

It is easy to see that the above equation is always equal to 0 for any election of the vectors X, Y, Z, and so φ is linearly integrable.

Thus we have the following result

Lemma 3.4. *If φ is an infinitesimal deformation belonging to H_5, then the law $L + \varphi$ will be a filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)$-color Lie superalgebra.*

4. Examples of filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)$-color Lie superalgebras via integrable deformations

In this section we are going to give a family of filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)$-color Lie superalgebras via linearly integrable deformations, i.e. a family of $\mathcal{F}^{n,m,p,t}$. Thus, we consider the symmetric bilinear map $h_{k,s}$ defined by the formula

$$h_{k,s}(Y_i, Y_i) = \begin{cases} X_s & \text{if } i = k \\ 0 & \text{in the other case} \end{cases}$$

with $1 \leq s \leq n$, $1 \leq k \leq m$ and satisfying the equation (5) of the Lemma 3.1 for $1 \leq i, j \leq m-1$. This equation rests

$$[X_0, h_5(Y_j, Y_k)] - h_5(Y_{j+1}, Y_k) - h_5(Y_{k+1}, Y_j) = 0$$

Thus, by induction the following formula for $h_{k,s}$ can be proved:

$$h_{k,s}(Y_i, Y_j) = (-1)^{k-i} \left(C_{j-k}^{k-i} - \frac{1}{2} C_{j-k-1}^{k-i} \right) X_{i+j+s-2k}$$

with $1 \leq i < j \leq m$, $k \leq \frac{i+j}{2}$. We suppose that $C_t^q = 0$ if $q < 0$ or $t < 0$ or $q > t$, and $C_0^0 = C_t^0 = 1$ with $t > 0$. In the remaining cases we have $C_t^q = \frac{t!}{q!(t-q)!}$.

We note that as $h_{k,s}$ verifies the equation (5) only for $1 \leq i, j \leq m-1$, then we have that $h_{k,s}$ is not always a cocycle of H_5. In particular, $h_{k,s}$ will be a cocycle of H_5 if and only if it satisfies the equation

$$[X_0, h_{k,s}(Y_i, Y_m)] - h_{k,s}(Y_{i+1}, Y_m) = 0, \text{ with } 1 \leq i \leq m.$$

It can be seen that $h_{k,s}$ verifies the above equation considering $s-2k \geq n-m-1$, i.e. if $s-2k \geq n-m-1$ then $h_{k,s} \in H_5 = Z^2(L; L) \cap \mathrm{Hom}(S^2 L_a, L_e)$. In fact, if $s-2k = n-m-1$, then

$$h_{k,s}(Y_1, Y_m) = (-1)^{k-1} \left(C_{m-k}^{k-1} - \frac{1}{2} C_{m-k-1}^{k-1} \right) X_n$$

and $h_{k,s}(Y_2, Y_m) = \cdots = h_{k,s}(Y_m, Y_m) = 0$ which clearly satisfy the above condition. Finally, if $s-2k > n-m-1$, then $h_{k,s}(Y_1, Y_m) = \cdots = h_{k,s}(Y_m, Y_m) = 0$ and also satisfies the above condition.

Recall that the model filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebra, $L^{n,m,p,t}$, is the simplest filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_i)$-color Lie superalgebra and it is defined in an adapted basis $\{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_p, W_1, \ldots, W_t\}$ by the following non-null bracket products

$$L^{n,m,p,t} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1 \\ [X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1 \\ [X_0, Z_k] = Z_{k+1} & 1 \leq k \leq p-1 \\ [X_0, W_s] = W_{s+1}, & 1 \leq s \leq t-1 \end{cases}$$

We observe that this definition does not depend on the election of the commutator factor β_i.

Therefore we have the following Lemma

Lemma 4.1. *The family of $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)$-color Lie superalgebras that follows*

$$\{L^{n,m,p,t} + h_{k,s}\}_{k,s}$$

with $s-2k \geq n-m-1$ is a family of filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_1)$-color Lie superalgebras.

Remark 4.2. It is not difficult to see that the above family is too a family of filiform $(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta_2)$-color Lie superalgebras.

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